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Some math about Particle Swarm Optimization

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For more information about the model itself see Jim Kennedy.

Analytical study

Iterative Representation versus Explicit Representation

In Particle Swarm Optimization, the usual iterative form is the following one:

$$\begin{cases} v_{t+1} = \alpha v_t + \beta\varphi y_t \\ y_{t+1} = -\gamma v_t + (\delta - \eta\varphi) y_t \end{cases} \quad \varphi \in \mathbf{R}^{+*} \quad t \in \mathbf{N}, \{y_t, v_t\} \in \mathbf{R}^2$$

Equ. 1

where $y_t = p - x_t$ (see also the Generalization section).

The matrix of the system is $M = \begin{bmatrix} \alpha & \beta\varphi \\ -\gamma & \delta - \eta\varphi \end{bmatrix}$.

By solving a classical second order differential equation, we find the explicit (analytic) representation (ER):

$$\begin{cases} v(t) = c_1 (\chi_1 e_1)^t + c_2 (\chi_2 e_2)^t \\ y(t) = \frac{1}{\beta\varphi} \left(c_1 (\chi_1 e_1)^t (\chi_1 e_1 - \alpha) + c_2 (\chi_2 e_2)^t (\chi_2 e_2 - \alpha) \right) \end{cases} \quad \varphi \in \mathbf{R}^{+*} \quad t \in \mathbf{N}, \{y(t), v(t)\} \in \mathbf{R}^2$$

Equ. 2

with

$$\begin{cases} e_1 = 1 - \frac{\varphi}{2} + \frac{\sqrt{\varphi^2 - 4\varphi}}{2} \\ e_2 = 1 - \frac{\varphi}{2} - \frac{\sqrt{\varphi^2 - 4\varphi}}{2} \end{cases} \text{ and } \begin{cases} c_1 = \frac{-\beta\varphi y(0) - (\alpha - \chi_2 e_2) v(0)}{\chi_2 e_2 - \chi_1 e_1} \\ c_2 = \frac{\beta\varphi y(0) + (\alpha - \chi_1 e_1) v(0)}{\chi_2 e_2 - \chi_1 e_1} \end{cases}$$

(see below for χ_1 and χ^2)
There is also an interesting algebraic representation wich take into account the fact that e_1 and e_2 are the eigenvalues of $\begin{bmatrix} 1 & \varphi \\ -1 & 1 - \varphi \end{bmatrix}$. We don't study it in this paper.

It is worth to note immediately an important difference between IR and ER: in the previous one t is always an integer and $v(t)$ and $y(t)$ are real numbers. In the second one we obtain real numbers if (and only if) t is an integer, but nothing prevent us to give any real positive value to t, and then $v(t)$

and $y(t)$ are "true" complex numbers. This fact will give us an elegant way to "explain" the system, by the use of a 5-dimensional space (see the "Particle swarm in Complexland " section).

From IR to ER

By computing the eigenvalues of the matrix M, we find

$$\begin{cases} e_1^{'} = \chi_1 e_1 = \frac{1}{2}\left(\alpha + \delta - \eta\varphi + \sqrt{(\eta\varphi)^2 - 4\beta\gamma\varphi + (\alpha - \delta)^2 + 2\eta\varphi(\alpha - \delta)}\right) \\ e_{21}^{'} = \chi_2 e_2 = \frac{1}{2}\left(\alpha + \delta - \eta\varphi - \sqrt{(\eta\varphi)^2 - 4\beta\gamma\varphi + (\alpha - \delta)^2 + 2\eta\varphi(\alpha - \delta)}\right) \end{cases} \quad \textbf{Equ. 3}$$

and then

$$\begin{cases} \chi_1 = \dfrac{\alpha + \delta - \eta\varphi + \sqrt{(\eta\varphi)^2 + 2\varphi(\alpha\eta - \delta\eta - 2\beta\gamma) + (\alpha - \delta)^2}}{2 - \varphi + \sqrt{\varphi^2 - 4\varphi}} \\ \chi_2 = \dfrac{\alpha + \delta - \eta\varphi - \sqrt{(\eta\varphi)^2 + 2\varphi(\alpha\eta - \delta\eta - 2\beta\gamma) + (\alpha - \delta)^2}}{2 - \varphi - \sqrt{\varphi^2 - 4\varphi}} \end{cases} \quad \textbf{Equ. 4}$$

This coefficients are always defined (but not necessarily real), for the denominator cannot be equal to zero.

Note 1

If we want χ_1 and χ_2 are real numbers for a given φ value, we must have some relations between the five real coefficients $\{\alpha, \beta, \gamma, \delta, \eta\}$. If we write the imaginary parts of χ_1 and χ_2 are equal to zero, we obtain

$$\begin{cases} \sqrt{|E|}\,(1 - sign(E))C - \left(\alpha + \delta - \eta\varphi + \frac{1}{2}\sqrt{|E|}\,(1 + sign(E))\right)\sqrt{B}(1 - A) = 0 \\ \sqrt{|E|}\,(1 - sign(E))C' - \left(\alpha + \delta - \eta\varphi - \frac{1}{2}\sqrt{|E|}\,(1 + sign(E))\right)\sqrt{B}(1 - A) = 0 \end{cases} \quad \textbf{Equ. 5}$$

with

$A = sign(\varphi^2 - 4\varphi)$

$B = |\varphi^2 - 4\varphi|$

$C = 2 - \varphi + \frac{1}{2}\sqrt{|\varphi^2 - 4\varphi|}\,(1 + sign(\varphi^2 - 4\varphi))$

$C' = 2 - \varphi - \frac{1}{2}\sqrt{|\varphi^2 - 4\varphi|}\,(1 + sign(\varphi^2 - 4\varphi))$

$D = C^2 + \frac{1}{4}|\varphi^2 - 4\varphi|\,(1 - sign(\varphi^2 - 4\varphi))^2$

$E = (\eta\varphi)^2 + 2\varphi(\alpha\eta - \delta\eta - 2\beta\gamma) + (\alpha - \delta)^2$

By combining them the two equalities of Equ. 5. Solutions are usually not completely independent of ϕ. To satisfy this equations,a set of possible conditions is

$$\begin{cases} E > 0 \\ A = -1 \ (\Leftrightarrow \phi < 4) \\ \alpha + \delta - \eta\phi = 0 \end{cases}$$

But this conditions are not necessary. For example, an interesting particular case (studied below) is $\alpha = \beta = \gamma = \delta = \eta = \chi \quad R_+^*$. We have then $\chi_1 = \chi_{\acute{E}} = \chi$ for any ϕ (Equ. 5 is always satisfied)

From ER to IR

From

we obtain

$$\begin{cases} \chi_1\left(2-\phi+\sqrt{\phi^2-4\phi}\right)+\chi_2\left(2-\phi-\sqrt{\phi^2-4\phi}\right)=2(\alpha+\delta-\eta\phi) \\ \chi_1\left(2-\phi+\sqrt{\phi^2-4\phi}\right)-\chi_2\left(2-\phi-\sqrt{\phi^2-4\phi}\right)=2\sqrt{(\eta\phi)^2+2\phi(\alpha\eta-\delta\eta-2\beta\gamma)+(\alpha-\delta)^2} \end{cases}$$

or

$$\begin{cases} 2(\alpha+\delta-\eta\phi)=(\chi_1+\chi_2)(2-\phi)+(\chi_1-\chi_2)\sqrt{\phi^2-4\phi} \\ 2\sqrt{(\eta\phi)^2+2\phi(\alpha\eta-\delta\eta-2\beta\gamma)+(\alpha-\delta)^2}=(\chi_1+\chi_2)\sqrt{\phi^2-4\phi}+(\chi_1-\chi_2)(2-\phi) \end{cases} \qquad \textbf{Equ. 6}$$

There are an infinity of solutions in $\{\alpha,\beta,\gamma,\delta,\eta\}$. We can add some others conditions. Let us study some particular classes of solutions.

Particular classes of solution

Class 1 model

$$\begin{cases} \alpha = \delta \\ \beta\gamma = \eta^2 \end{cases} \qquad \textbf{Equ. 7}$$

In this particular case, From the Equ. 4 we obtain

$$\begin{cases} \alpha = \dfrac{1}{4}\left(2(\chi_1+\chi_2)+(\chi_1-\chi_2)\left(\sqrt{\phi^2-4\phi}+\phi\dfrac{2-\phi}{\sqrt{\phi^2-4\phi}}\right)\right) \\ \eta = \dfrac{1}{2}\left(\chi_1+\chi_2+\dfrac{2-\phi}{\sqrt{\phi^2-4\phi}}(\chi_1-\chi_2)\right) \end{cases}$$

A easy way to be sure to obtain real coefficients is then to have $\chi_1 = \chi_2 = \chi \quad \mathbf{R}$.Under this additional condition, a class of solution is simply given by

$$\boxed{\alpha = \beta = \gamma = \delta = \eta = \chi}$$ **Equ. 8**

Class 1' model

$$\begin{cases} \alpha = \beta \\ \gamma = \delta = \eta = 1 \end{cases}$$ **Equ. 9**

From Equ. 4 we obtain

$$\alpha = \frac{(\chi_1 + \chi_2)(2-\phi) + (\chi_1 - \chi_2)\sqrt{\phi^2 - 4\phi}}{2} + \phi - 1$$

If we add again the condition $\chi_1 = \chi_2 = \chi \quad \mathbf{R}$, we find

$$\boxed{\alpha = \chi(2-\phi) + \phi - 1}$$ **Equ. 10**

I we don't add this condition, we have nevertheless from the Equ. 4

$$\begin{cases} \chi_1 = \dfrac{\alpha + 1 - \phi + \sqrt{\phi^2 + 2\phi(\alpha - 3) + (\alpha - 1)^2}}{2 - \phi + \sqrt{\phi^2 - 4\phi}} \\ \chi_2 = \dfrac{\alpha + 1 - \phi - \sqrt{\phi^2 + 2\phi(\alpha - 3) + (\alpha - 1)^2}}{2 - \phi + \sqrt{\phi^2 - 4\phi}} \end{cases}$$

Class 1'' model

$$\alpha = \beta = \gamma = \eta$$ **Equ. 11**

$$\alpha = \frac{2\delta + (\chi_1 + \chi_2)(\phi - 2) - (\chi_1 - \chi_2)\sqrt{\phi^2 - 4\phi}}{2(\phi - 1)}$$ **Equ. 12**

For « historical » reason and for the its simplicity, the case $\delta = 1$ has been well studied.

Class 2 model

$$\begin{cases} \alpha = \beta = 2\delta \\ \eta = 2\gamma \end{cases}$$ **Equ. 13**

We have then

$$\begin{cases} 2(3\delta - 2\gamma\phi) = (\chi_1 + \chi_2)(2-\phi) + (\chi_1 - \chi_2)\sqrt{\phi^2 - 4\phi} \\ 2|2\gamma\phi - \delta| = (\chi_1 + \chi_2)\sqrt{\phi^2 - 4\phi} + (\chi_1 - \chi_2)(2-\phi) \end{cases}$$

which give us γ and δ.

Again, an easy way to obtain real coefficients for every ϕ value is to have $\chi_1 = \chi_2 = \chi$. Then we have

$$\begin{cases} 3\delta - 2\gamma\phi = \chi(2-\phi) \\ |2\gamma\phi - \delta| = \chi\sqrt{\phi^2 - 4\phi} \end{cases}$$

In the case $2\gamma\phi \geq \delta$ we obtain

$$\boxed{\begin{cases} \delta = \chi\dfrac{2-\phi+\sqrt{\phi^2-4\phi}}{2} = \chi e_1 \\ \gamma = \chi\dfrac{2-\phi+3\sqrt{\phi^2-4\phi}}{4\phi} \end{cases}} \qquad \textbf{Equ. 14}$$

It is interesting to note (it will be useful to study the convergence) that we have

- for the Class 1 model, with the condition $\chi_1 = \chi_2 = \chi$

$$\begin{cases} |e_1^{'}| = \chi|e_1| \\ |e_2^{'}| = \chi|e_2| \end{cases} \qquad \textbf{Equ. 15}$$

- for the Class 1' model, with the condition $\chi_1 = \chi_2 = \chi$ and for $\phi \leq 2$

$$\begin{cases} |e_1^{'}| = \left|\chi\left(1-\dfrac{\phi}{2}\right)+\dfrac{\sqrt{\chi^2\left(4-4\phi+\phi^2\right)+4\chi(\phi-2)+4(\phi-1)}}{2}\right| \leq \chi|e_1| \\ |e_2^{'}| = \left|\chi\left(1-\dfrac{\phi}{2}\right)-\dfrac{\sqrt{\chi^2\left(4-4\phi+\phi^2\right)+4\chi(\phi-2)+4(\phi-1)}}{2}\right| \leq \chi|e_2| \end{cases} \qquad \textbf{Equ. 16}$$

- for the the Class 2 model

$$\begin{cases} |e_1^{'}| = \chi\left|\dfrac{3}{2}-\dfrac{3}{4}\phi+\dfrac{3}{4}\Delta-\dfrac{1}{2}\phi^2+\dfrac{1}{4}\phi^3-\dfrac{3}{4}\phi^2\Delta+\dfrac{1}{4}\left|2-\phi-2\phi^2-2\phi^3+\Delta-3\phi^2\Delta\right|\right| = \chi\left|e_{1,class2}\right| \\ |e_2^{'}| = \chi\left|\dfrac{3}{2}-\dfrac{3}{4}\phi+\dfrac{3}{4}\Delta-\dfrac{1}{2}\phi^2+\dfrac{1}{4}\phi^3-\dfrac{3}{4}\phi^2\Delta-\dfrac{1}{4}\left|2-\phi-2\phi^2-2\phi^3+\Delta-3\phi^2\Delta\right|\right| = \chi\left|e_{2,class2}\right| \end{cases} \qquad \textbf{Equ. 17}$$

with $\Delta = \sqrt{\phi^2 - 4\phi}$

As we will see below in the Convergence and Space of States sections, it means that for this cases, we will just have to choose

$\chi < \dfrac{1}{|e_2|}$, $\chi < \dfrac{1}{|e_2|}$ and $\phi \leq 2$, $\alpha < \dfrac{1}{|e_2|}$, $\chi < \left|e_{2,class2}\right|$

respectively, to have a convergent system.

Particle in Complexland

Back to reality

Removing the discontinuity

The system has usually has a discontinuity in ϕ due to fact that there is the term $\sqrt{(\eta\phi)^2 - 4\beta\gamma\phi + (\alpha-\delta)^2 + 2\eta\phi(\alpha-\delta)}$ in the eigenvalues.

So, if we want to have a completely continuous system, we just have to choose $\{\alpha,\beta,\gamma,\delta,\eta\}$ so that

$$\begin{cases} \{\alpha,\beta,\gamma,\delta,\eta\} \quad \mathbf{R}^5 \\ \quad \phi \quad \mathbf{R}^+ (\eta\phi)^2 - 4\beta\gamma\phi + (\alpha-\delta)^2 + 2\eta\phi(\alpha-\delta) \geq 0 \end{cases}$$

By computing the discriminant we find the last condition is equivalent to

$$\beta\gamma\left(-\beta\gamma + \eta(\alpha-\delta)\right) > 0$$

In order to be "physically plausible", we are looking for positive parameters $\{\alpha,\beta,\gamma,\delta,\eta\}$. So the conditionbecomes

$$\boxed{\beta\gamma < \eta(\alpha-\delta)}$$ **Equ. 18**

This conditions specify a "volume" in $\mathbf{R}^4$ for the admissible values of the parameters..

Removing the imaginary

By using the above condition the trajectory is usually still partly in a complex space, as soon as one of the eigenvalue is negative (due to the fact that $(-1)^t$ is a complex if t is not an integer). So we may want to find some stronger conditions in order to have always positive eigenvalues.
By noting that we have

$$\begin{cases} e_1 > 0 \\ e_2 > 0 \end{cases} \Leftrightarrow \begin{cases} e_1 + e_2 > 0 \\ e_1 e_2 > 0 \end{cases}$$

we find easily

$$\boxed{\begin{cases} \alpha(\delta - \eta\phi) + \gamma\beta\phi > 0 \\ \alpha + \delta - \eta\phi > 0 \end{cases}}$$ **Equ. 19**

Note 2

From an algebraic point of view, these conditions can be written as

$$\begin{cases} \det(M) > 0 \\ trace(M) > 0 \end{cases}$$

But now this conditions are depending on ϕ. Nevertheless, if we know the maximun ϕ value, we can rewrite them

$$\boxed{\begin{cases} \dfrac{\alpha\delta}{\alpha\eta-\gamma\beta} > \phi_{max} \\ \dfrac{\alpha+\delta}{\eta} > \phi_{max} \end{cases}} \qquad \textbf{Equ. 20}$$

Under this conditions, the system is completely real.

An under the conditions Equ. 19 $\boxed{\beta\gamma < \eta(\alpha-\delta)}$ and $\boxed{\begin{cases} \dfrac{\alpha\delta}{\alpha\eta-\gamma\beta} > \phi_{max} \\ \dfrac{\alpha+\delta}{\eta} > \phi_{max} \end{cases}}$ Equ. 20, the system is continuous *and* real.

Example

If we suppose $\alpha = \beta = 1$ and $\delta = \eta$, the conditions become

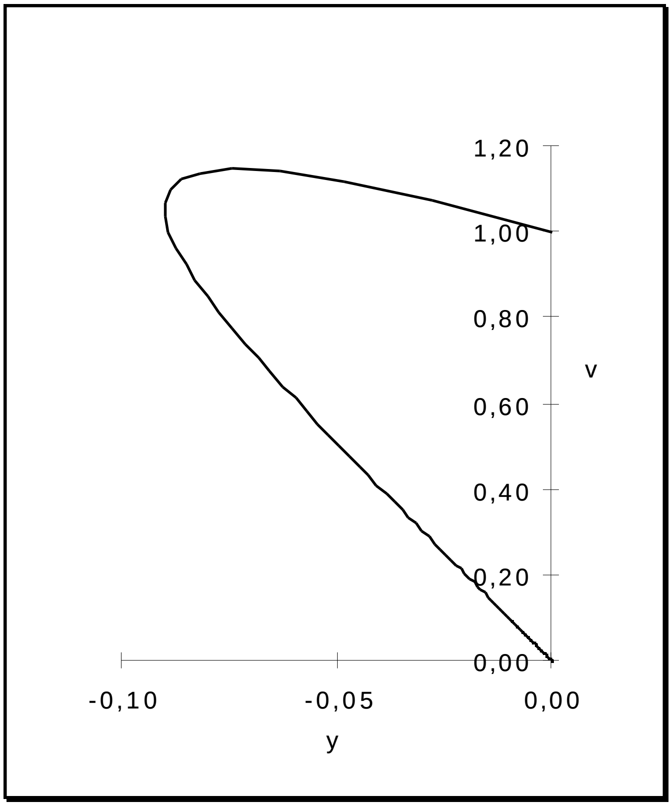
$$\begin{cases} \delta < \dfrac{1}{\phi_{max}} \\ \dfrac{\delta(\phi_{max}-1)}{\phi_{max}} < \gamma < \delta(1-\delta) \end{cases}$$

For example

$$\begin{cases} \phi_{max} = 10 \\ y_0 = 0, v_0 = 1 \\ \alpha = \beta = 1 \\ \gamma = \dfrac{1}{2}\left(\dfrac{\delta(\phi_{max}-1)}{\phi_{max}} + \delta(1-\delta)\right) = 0.08915 \\ \delta = \eta = \dfrac{0.99}{\phi_{max}} = 0.099 \end{cases}$$

The system converges quit quickly (about 25 time steps) and at each time step the values of y and v are almost the same, for a large range of ϕ values. The Figure 1 shows the result for $\phi = 4$.

Figure 1. $\phi = 4$



Reality and convergence

The quick convergence of the above example suggests an interesting question. Does "reality" implies convergence ? Or, in other terms, we are wondering if we have

$$\begin{cases} \dfrac{\alpha\delta}{\alpha\eta - \gamma\beta} > \phi_{max} \\ \dfrac{\alpha + \delta}{\eta} > \phi_{max} \end{cases} \qquad \begin{cases} \left|e_1^{'}\right| < 1 \\ \left|e_2^{'}\right| < 1 \end{cases}$$

Unfortunately the answer is negative.

Example

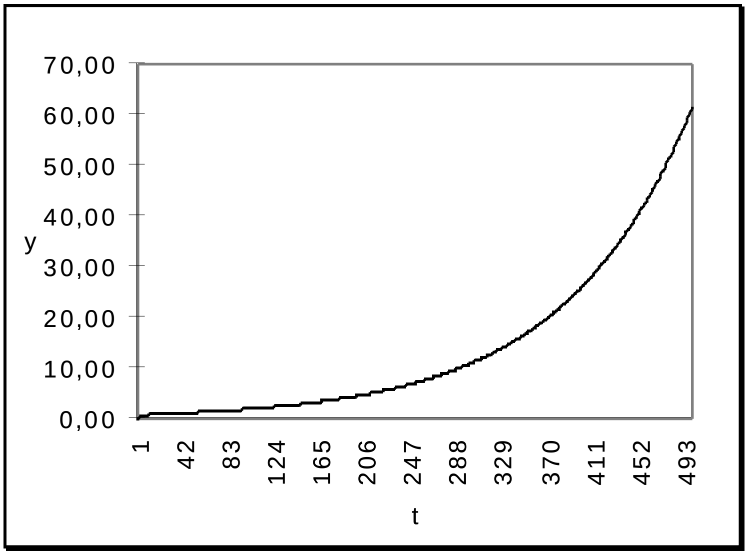
$$\begin{cases} \phi_{max} = 10 \\ y_0 = 0, v_0 = 1 \\ \alpha = \beta = 1.1 \\ \gamma = 0.0891495 \\ \delta = \eta = 0.099 \end{cases}$$

We have indeed

$$\begin{cases} \dfrac{\alpha\delta}{\alpha\eta - \gamma\beta} = 10.05 > \phi_{max} \\ \dfrac{\alpha + \delta}{\eta} = 12.11 > \phi_{max} \end{cases}$$

but for $\phi = 0.1$ (for instance) we obtain $|e_1^{'}| = 1.09$ and the system diverges (see Figure 2).

Figure 2. "Reality" doesn't imply convergence.



Convergence and Space of States

From the Equ. 15 and the Equ. 3 we find the criterion of convergence:

$$\boxed{\begin{cases} |e_1^{'}| < 1 \\ |e_2^{'}| < 1 \end{cases}} \qquad \textbf{Equ. 21}$$

In the explicit general form of the system, v_t and y_t are usually "true" complex numbers. So, the whole system should be represented in a 5-dimension space $(\mathrm{Re}(y), \mathrm{Im}(y), \mathrm{Re}(v), \mathrm{Im}(v), \phi)$.

Here we study more completelysome examples of an important class of constricted cases : the onez with just one *constriction coefficient* ,

Constriction for model Type 1

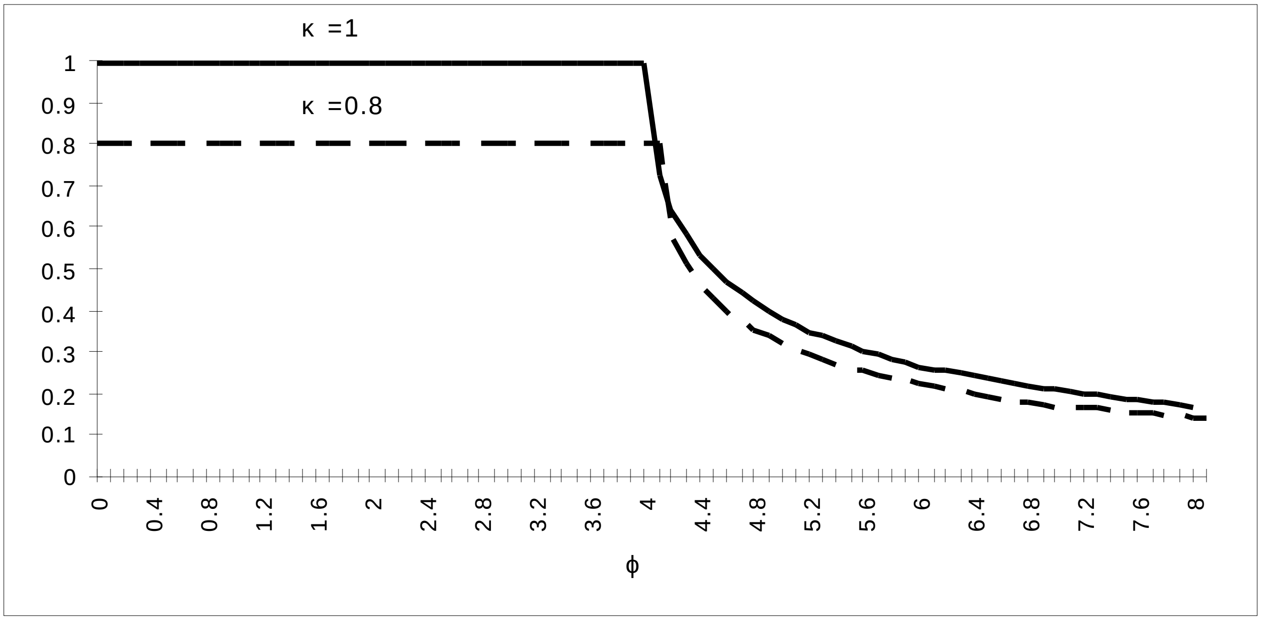
We use the implicit representation of the model class 1

$$\boxed{\begin{cases} v_{t+1} = \chi(v_t + \phi y_t) \\ y_{t+1} = -\chi\big(v_t + (1-\phi)y_t\big) \end{cases}}$$

From the Equ. 15 we know that the convergence criterion is satisfied if we have $\chi < \min\left(\frac{1}{|e_1|}, \frac{1}{|e_2|}\right)$. As $|e_1| \le |e_2|$ we can take as constriction coefficient

$$\boxed{\chi = \frac{\kappa}{|e_2|}, \kappa \quad]0,1[} \qquad \textbf{Equ. 22}$$

Figure 3. Constriction coefficient for model Type 1



Constriction for model Type 1'

We use the following implicit representation (with χ instead of α)

$$\boxed{\begin{cases} v(t+1) = \chi(v(t) + \varphi y(t)) \\ y(t+1) = -v(t) + (1-\varphi)y(t) \end{cases}}$$

We can take again

$$\boxed{\chi = \frac{\kappa}{|e_2|}, \kappa \quad]0,1[\text{, for } \varphi \quad]0,2[} \qquad \textbf{Equ. 23}$$

but we have seen this formula is a priori valid only for φ<2, so it is interesting to find directly another constriction coefficient. We have here

$$e_2^{'} = \frac{\chi + 1 - \varphi}{2} - \frac{\sqrt{(\chi - 1)^2 + \varphi^2 - 2\varphi - 2\chi\varphi}}{2}$$

The expression under the square root is negative for $\chi \quad]1+\varphi - 2\sqrt{\varphi}, 1+\varphi + 2\sqrt{\varphi}[$. In this case, the eigenvalue is a "true" complex number, and we have simply $|e_2^{'}| = \sqrt{\chi}$. So, if $1+\varphi - 2\sqrt{\varphi} < 1$, that is to say if $\varphi<4$, we just have to choose $\chi \quad]1+\varphi-2\sqrt{\varphi},1[$ to satisfy the convergence criterion. So, for example, we can define

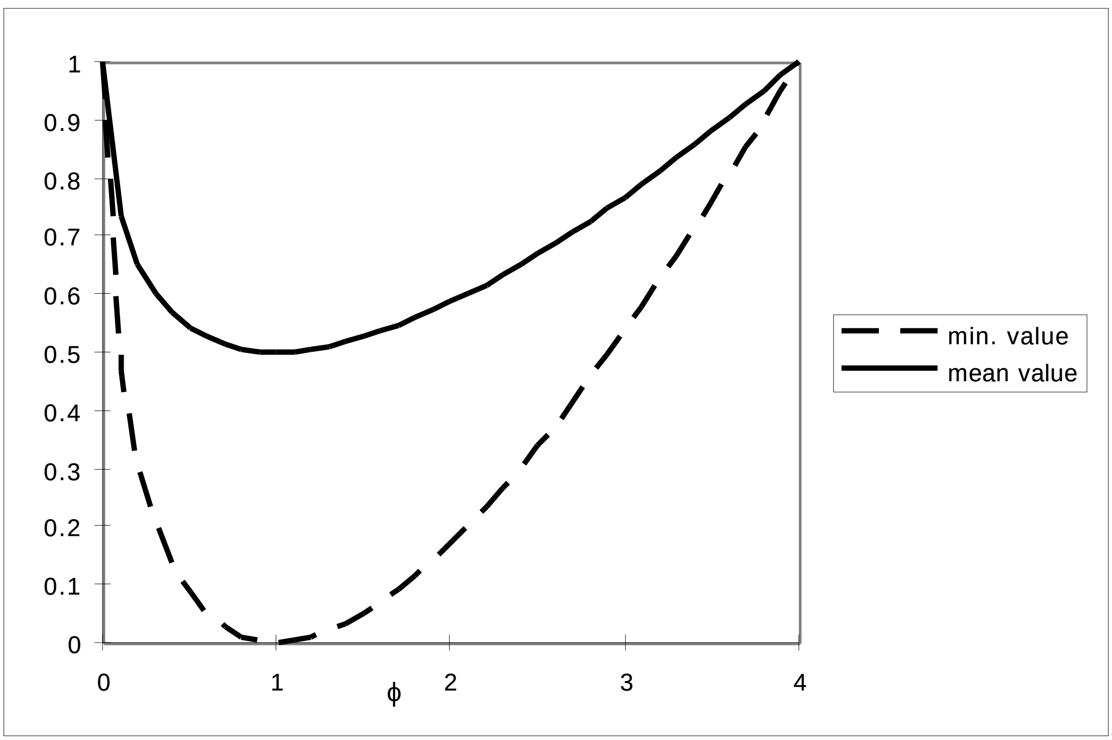
$$\boxed{\chi=\frac{2+\varphi-2\sqrt{\varphi}}{2}\text{, for } \varphi \quad]0,4[} \qquad \textbf{Equ. 24}$$

Now, can we find another formula for greater φ values ? The answer is "No". For in this case, $e_2^{'}$ is a real number, on its absolute value is

- strictly decreasing on $\alpha \quad]0,1+\varphi-2\sqrt{\varphi}]$, and the minimal value is $\sqrt{\varphi}-1$ (greater than 1),
- strictly decreasing on $\alpha \quad [1+\varphi+2\sqrt{\varphi},\infty[$, and its limit is 1.

For simplicity, we may want to have the same formula than for the Type 1, not only for $\varphi<2$, but also for $\varphi<4$. This is indeed also possible, but then κ can not be too small, depending of φ. More precisely, we must have $\kappa>\left(1+\varphi-2\sqrt{\varphi}\right)|e_2|$. But as for $\varphi<4$, we have $|e_2|=1$, it just means that the curves in the Figure 4can then by interpreted as the mean (resp. Minimal) acceptable κ value for sure convergence. For example, for $\varphi=3$, we must have $\kappa>0.536$. In particular, we can note there is no restriction on $\kappa \quad]0,1[$ if $\varphi=1$.

Figure 4. Constriction coefficient for convergence Type 1'.



Constriction type 1''

Referring to the Class 1'' model, in the particular case $\delta=1$, we use the following implicit representation (with χ instead of α)

$$\boxed{\begin{cases} v(t+1) = \chi\left(v(t) + \varphi y(t)\right) \\ y(t+1) = -\chi v(t) + (1-\chi\varphi)\, y(t) \end{cases}}$$

In fact, this system is just a transformation of the almost « classical » one

$$\begin{cases} v(t+1) = \chi\left(v(t) + \varphi\left(p - x(t)\right)\right) \\ x(t+1) = v(t+1) + x(t) \end{cases}$$

so it may be interesting to detail here how, in practice, the constriction coefficient is found and proved.

Step 1. Matrix of the system

We have immediately $M = \begin{bmatrix} \chi & \chi\varphi \\ -\chi & 1-\chi\varphi \end{bmatrix}$.

Step 2. Eigenvalues

They are the two solutions of the equation

$$Z^2 - trace(M)Z + determinant(M) = 0$$

that is to say of

$$Z^2 - \left(\chi + 1 - \chi\varphi\right)Z + \chi = 0$$

We find here

$$\begin{cases} e_1' = \dfrac{\chi + 1 - \chi\varphi + \sqrt{\Delta}}{2} \\ e_2' = \dfrac{\chi + 1 - \chi\varphi - \sqrt{\Delta}}{2} \end{cases}$$

with

$$\begin{aligned} \Delta &= trace(M)^2 - 4determinant(\mathrm{M}) \\ &= \chi^2\left(\varphi^2 - 4\varphi + 2\varphi\left(1 - \frac{1}{\chi}\right) + \left(1 - \frac{1}{\chi}\right)^2\right) \end{aligned}$$

Step 3. Complex and real areas on φ

The discriminant Δ is negative for the φ values $\left]1 + \frac{1}{\chi} - \frac{2}{\sqrt{\chi}}, 1 + \frac{1}{\chi} + \frac{2}{\sqrt{\chi}}\right[$. In this area the eigenvalues are true complex numbers and their absolute value (i;e. module) is simply $\sqrt{\chi}$.

Step 4. Extension of the complexe area and constriction coefficient

In the complex area, according to the convergence criterion, we just have to choose $\chi < 1$. So the idea is to find a constriction coefficient depending on φ so that the eigenvalues are true complex numbers for a large field of φ values , for in this case the common absolute value of the eigenvalues is simply

$$\begin{cases} \sqrt{\dfrac{2\kappa}{\varphi - 2 + \sqrt{\varphi^2 - 4\varphi}}} & \text{for } \varphi \geq 4 \\ \sqrt{\kappa} & \text{either} \end{cases}$$ **Equ. 25**

which is smaller than 1 for all φ values as soon as κ is itself smaller than 1.

This is generally the most difficult step and needs sometimes some intuition. But here, we may remark three points:

- the determinant of the matrix is equal to χ,
- this is the same as in Constriction Type 1,
- we know from the algebraic point of view the system is (eventually) convergent « like » M^T

So it is very probable that the same constriction coefficient as for Type 1 should work. We try then

$$\boxed{\chi = \frac{\kappa}{|e_2|}, \kappa \in]0,1[}$$ **Equ. 26**

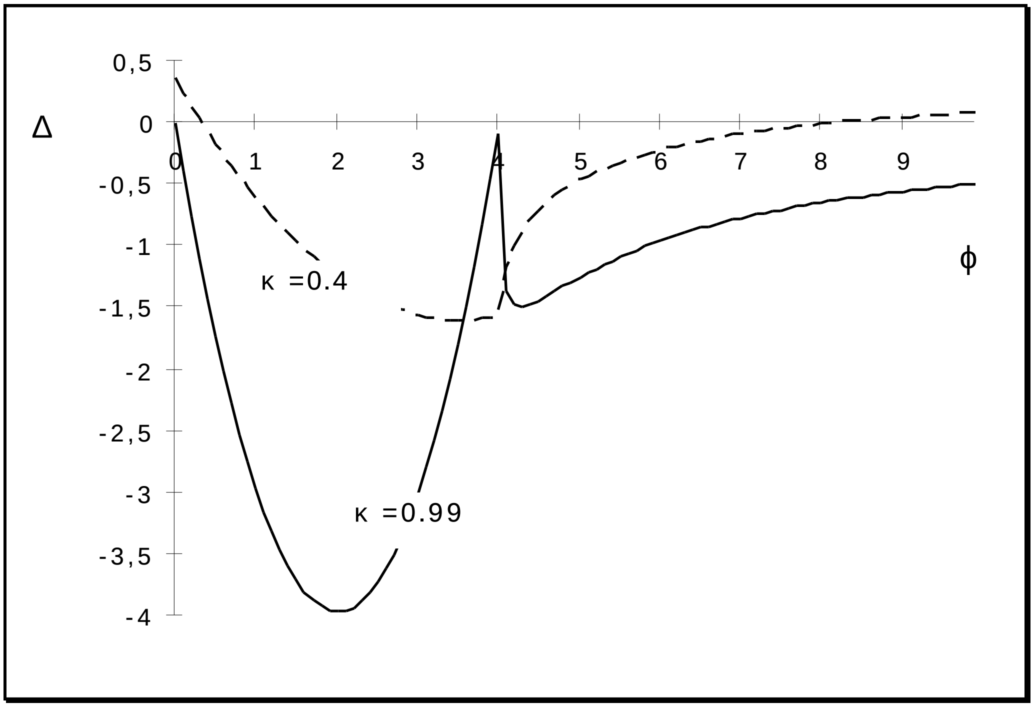
that is to say $\chi = \begin{cases} \dfrac{2\kappa}{\varphi - 2 + \sqrt{\varphi^2 - 4\varphi}} & \text{for } \varphi \geq 4 \\ \kappa & \text{either} \end{cases}$

It is easy to see that Δ is negative only between two φ values φ_{min} and φ_{max} depending on κ. The general algebraic form of φ_{max} is quite complicated (polynom in κ^6 with some coefficients being roots of an equation in κ^4) so it is much easier to compute it indirectly for some κ values. If we suppose φ_{min} smaller than 4, we have $\chi = \kappa$ and by solving $\Delta = 0$ we find simply $\varphi_{min} = \dfrac{\kappa^2 + \kappa - 2\kappa^{3/2}}{\kappa^2}$, which is then valid as soon as $\kappa \geq \dfrac{1}{9}$.

The Figure 5 shows how the discriminant is depending on φ, for two κ values. It is negative between the φ values given below:

κ	φ_{min}	φ_{max}
0.4	0.3377	8.07
0.99	0.000025	39799.76

Figure 5. Discriminant Δ versus φ.



So, in practice, the constriction coefficient works for almost φ values as soon as κ is near of 1. If κ is exactly equal to 1, we have $\varphi_{min} = 0$ and $\varphi_{max} = \infty$. Theoretically the convergence is not completely sure , for we have then $|e_1'| = |e_2'| = 1$ but it appears in the simulations we still have convergence, due to the fact we are using several particles and random φ values.

Moderate constriction

We use the following explicit representation:

$$\begin{cases} v(t) = c_1 e_1^t + c_2 (\chi e_2)^t \\ y(t) = \dfrac{1}{\varphi}\left(c_1 e_1^t (e_1 - 1) + c_2 (\chi e_2)^t (\chi e_2 - 1)\right) \end{cases}$$
$$\chi = \frac{\kappa}{|e_2|}, \kappa \quad]0,1[$$

that is to say $\begin{cases} \chi_1 = 1 \\ \chi_2 = \chi \end{cases}$. From the Equ. 6 we obtain

$$\begin{cases} 2(\alpha + \delta - \eta\varphi) = (1+\chi)(2-\varphi) + (1-\chi)\sqrt{\varphi^2 - 4\varphi} \\ 2\sqrt{(\eta\varphi)^2 + 2\varphi(\alpha\eta - \delta\eta - 2\beta\gamma) + (\alpha - \delta)^2} = (1+\chi)\sqrt{\varphi^2 - 4\varphi} + (1-\chi)(2-\varphi) \end{cases}$$

There are an infinity of possibilities for the parameters α..η, that is to say there are an infinity of different implicit representations which give this same explicit one. For example:

$$\begin{cases} \alpha = \dfrac{\phi + 2\chi - \chi\phi}{2} + \dfrac{\sqrt{\phi^2 - 4\phi}}{2}(1-\chi) \\ \beta = -\dfrac{1}{2\phi}\left(\phi - 3\chi\phi - \phi^2 + \chi\phi^2 + \sqrt{\phi^2 - 4\phi}\,(1 + \chi\phi - \chi - \phi)\right) \\ \gamma = \delta = \eta = 1 \end{cases}$$

or

$$\begin{cases} \alpha = \beta = 1 \\ \gamma = \dfrac{\phi(1+\chi) - \sqrt{\phi^2 - 4\phi}\,(1-\chi)}{2\phi} \\ \delta = \eta = \dfrac{\phi + \chi(\phi - 2) - \sqrt{\phi^2 - 4\phi}\,(1-\chi)}{2(\phi - 1)} \end{cases}$$

From a mathematical point of view, this case is "richer" than the previous ones, for we have no more explosion, but not always convergence either. That is why we will study it more in detail. We could call this kind of system "stabilized", for, as we will see, the representative point in the state space tends to move along an "attractor" which is not always reduced to a single point, as in classical convergence.

The Figure 6 and its sections show what it is usually studied, that is to say just the "real" restrictions $\left(\mathrm{Re}(y),\mathrm{Re}(v),\phi\right)$. We can clearly see the three cases:

- "spiral" easy convergence towards a non trivial attractor for $\phi<4$ (Figure 7),
- difficult convergence for $\phi=4$ (Figure 8),
- quick almost linear convergence for $\phi>4$ (Figure 9).

Figure 6. Surface (Re(*y*), Re(*v*), ϕ), *t*=0..50, ϕ=0..20, *y*(0)=0, *v*(0)=1, κ=0.8

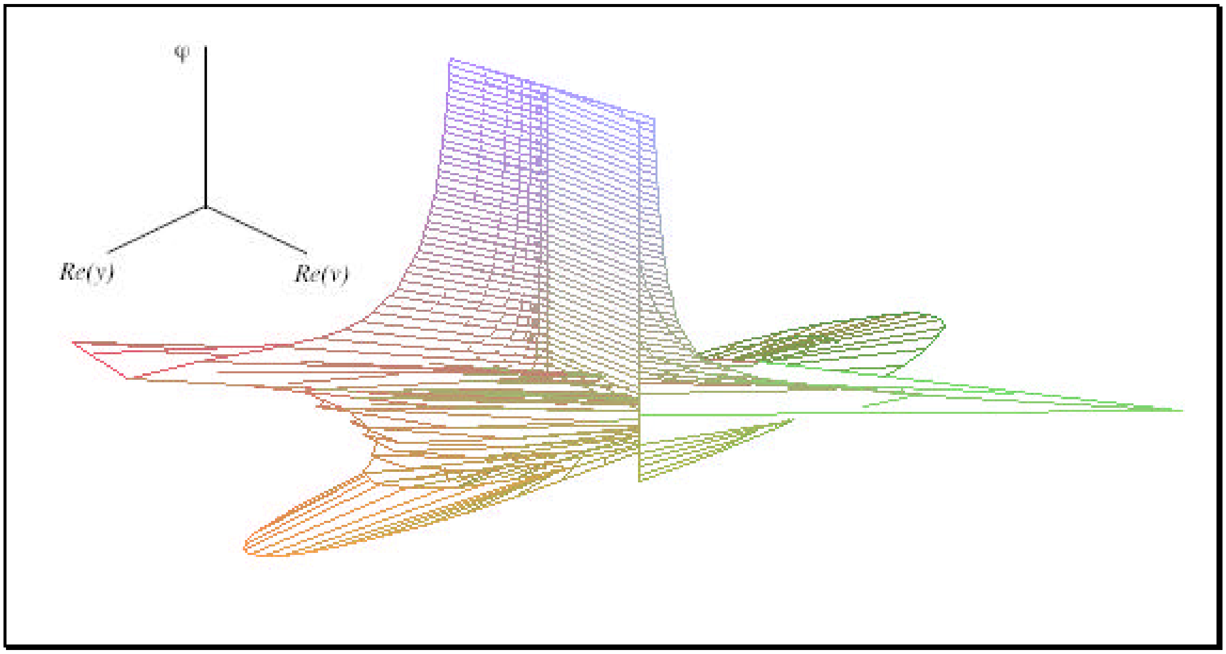


Figure 7. Section ϕ=2.5

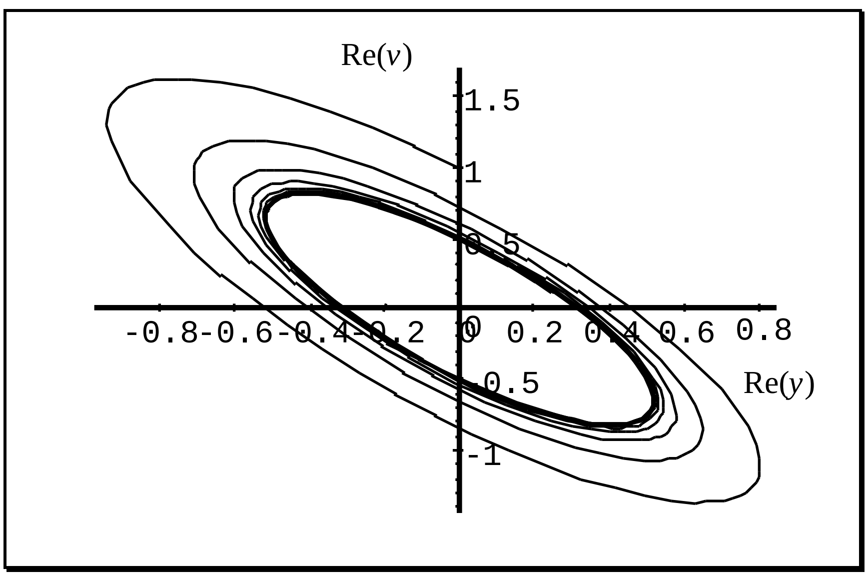


Figure 8. Section ϕ=4

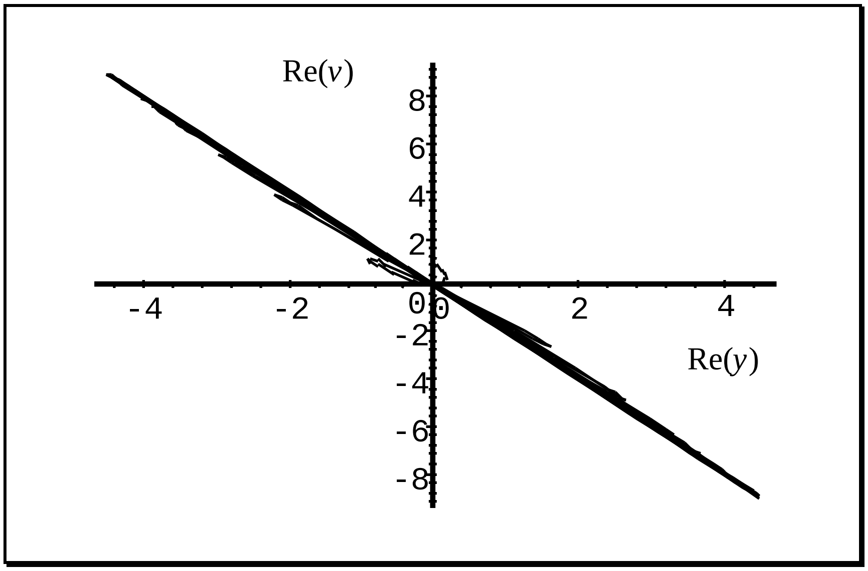
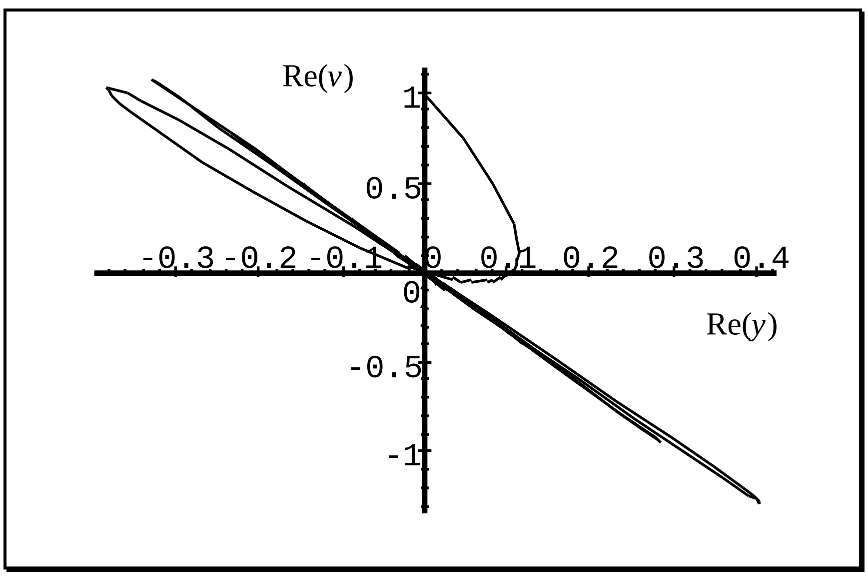


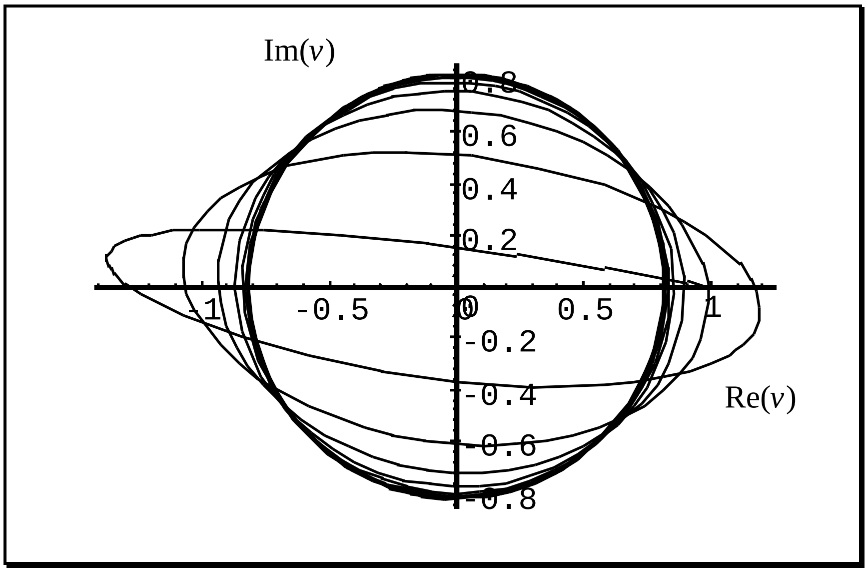
Figure 9. Section ϕ=6



Attractors and convergence

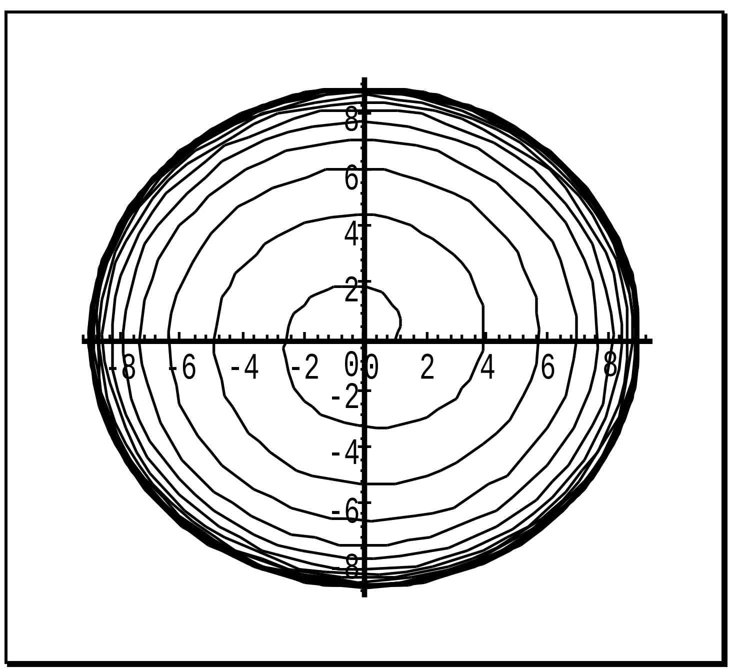
Nevertheless, it is interesting to have a look at what is the "true" system. The following figures show some others sections of the whole surface in R^5.

Figure 10. $\left(\mathrm{Re}(v),\mathrm{Im}(v)\right)$**, ϕ=2.5.**



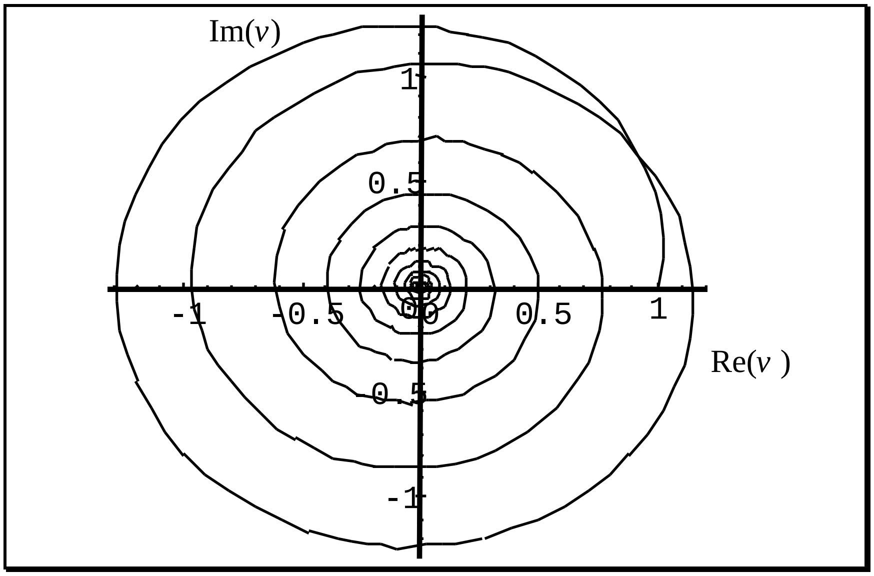
Attractor: circle center (0,0) radius=0.819

Figure 11. $\left(\mathrm{Re}(v),\mathrm{Im}(v)\right)$**, ϕ=4.**



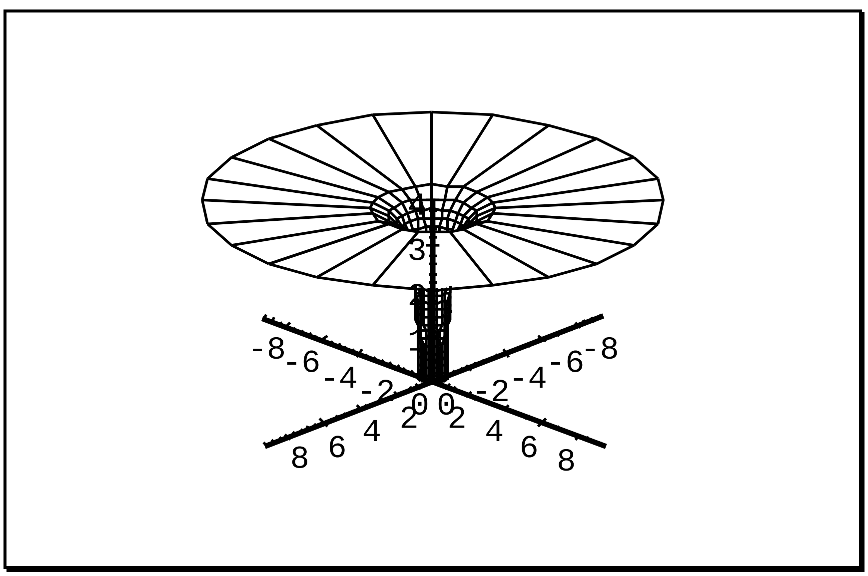
Attractor: circle center (0,0) radius=9

Figure 12. $(\mathrm{Re}(v), \mathrm{Im}(v))$, **ϕ=6.**



Attractor: circle center (0,0) radius=0

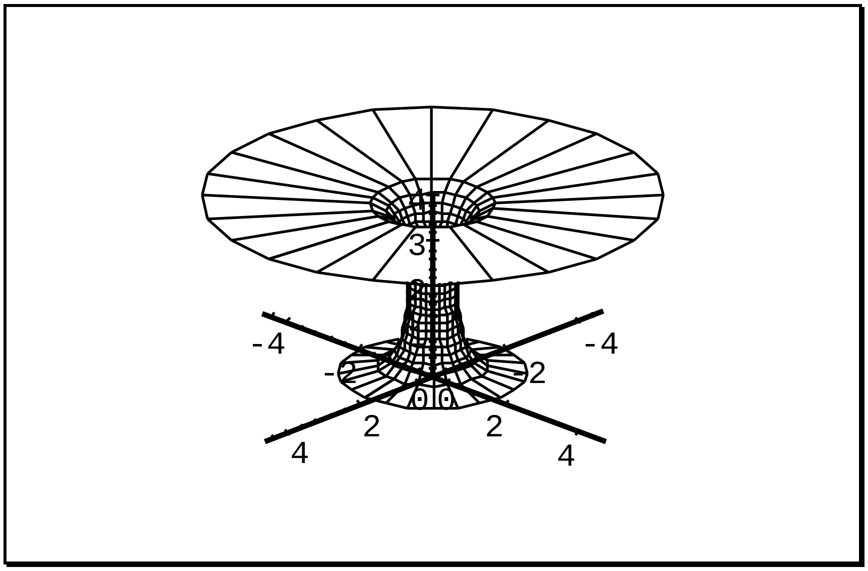
Figure 13. Global attractor for *v* and ϕ ≤ 4. Axis (Re(v), Im(v), ϕ), κ=0.8



Note 3

There is a discontinuity, for the radius is equal to zero for ϕ>4.

Figure 14. Global attractor for *y* and $\varphi \leq 4$. Axis (Re(y), Im(y), φ), κ=0.8



So, what it seems to be an "oscillation" in the real world is in fact a continuous spiralic move in a complex space. More important, the attractor is very easy to define: it is the "circle" $c_1 e_1^t$ (center (0,0) and radius ρ). So when $\varphi < 4$, $\rho = |c_1 e_1|$ and when φ is greater than 4 $\rho = 0$ ($\lim_{t \to \infty} |c_1 e_1{}^t|$ with $|e_1| < 1$) for the constriction coefficient χ has been precisely chosen so that the part $c_2 (\chi e_2)^t$ of $v(t)$ tends to zero. This gives us a good and simple intuitive way to transform this stabilization into a true convergence. We just have to use a second coefficient to reduce the attractor, in the case $\varphi \leq 4$, so that $e_1 \to \chi' e_1, \chi' \leq \frac{\kappa'}{|e_1|}, \kappa' \quad]0,1[$

Note 4

As we are studying here the "one constriction coefficient models", we have to choose $\chi' = \chi$, and finally we retrieve the type 1 constriction. But now, we understand better *why* it works.

Generalization

We study here the more general system defined by

$$\begin{cases} v(t+1) = v(t) + \varphi_1 (p_1 - x(t)) + \varphi 2(p2 - x(t)) \\ x(t+1) = v(t+1) + x(t) \end{cases}$$

We just have to define

$\varphi = \varphi_1 + \varphi_2$

$$p = \frac{\varphi_1 p_1 + \varphi_2 p_2}{\varphi_1 + \varphi_2}$$

$y(t) = p - x(t)$

to obtain exactly the same system as the one studied above.

For instance, if we have a cycle for $\varphi = \varphi_c$, so we have an infinity of cycles for the values $\{\varphi_1, \varphi_2\}$ so that $\varphi_1 + \varphi_2 = \varphi_c$.

If we compute the constriction coefficient, we obtain

$$\chi = \frac{\kappa}{|e_2|} = \frac{\kappa}{\left|1 - \frac{\varphi}{2} - \frac{\sqrt{\varphi(\varphi-4)}}{2}\right|} = \frac{2\kappa}{\left|2 - \varphi - \sqrt{\varphi(\varphi-4)}\right|}$$

$$= \frac{2\kappa}{\left|2 - \varphi_1 - \varphi_2 - \sqrt{(\varphi_1+\varphi_2)(\varphi_1+\varphi_2-4)}\right|}, \text{ if } (\varphi_1+\varphi_2) > 4$$

$= \kappa$ else

$\kappa \quad]0,1[$

Coming back to the (v,x) system, we have then

$$\begin{cases} v(t+1) = v(t) + \varphi_1(p_1 - x(t)) + \varphi 2(p2 - x(t)) \\ x(t+1) = \chi v(t+1) + \chi x(t) + (1-\chi)\dfrac{\varphi_1 p_1 + \varphi_2 p_2}{\varphi_1 + \varphi_2} \end{cases}$$

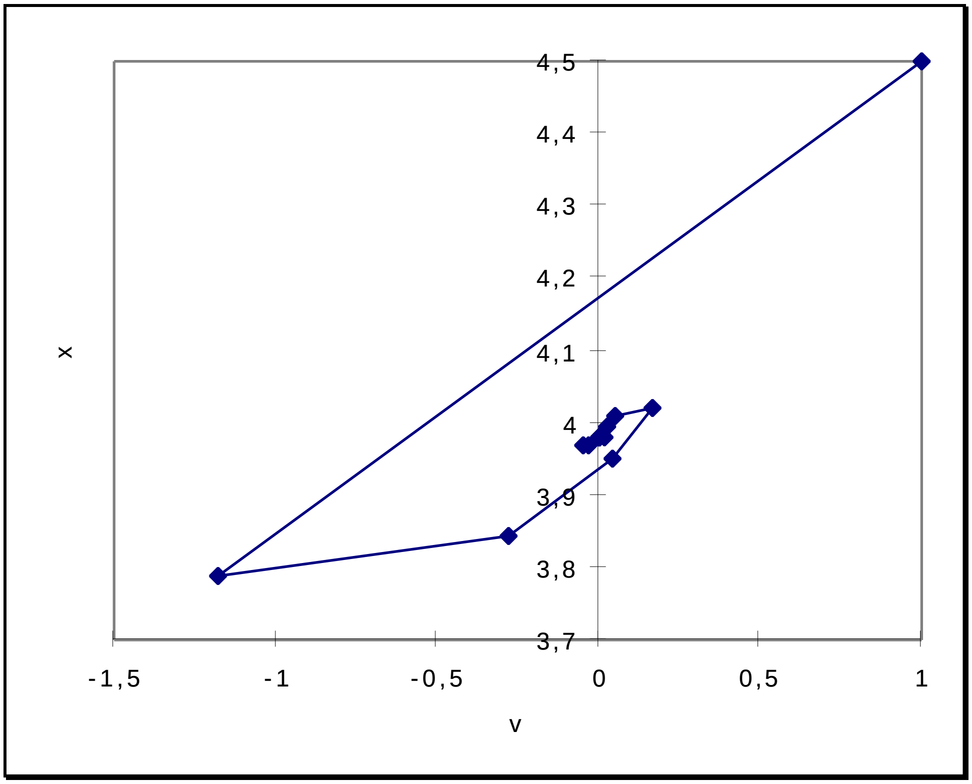
The use of the constriction coefficient could be seen as a recommendation to the particle "Make more little steps"

The convergence is towards the point $(v = 0, x = \frac{\varphi_1 p_1 + \varphi_2 p_2}{\varphi_1 + \varphi_2})$. Remember v is in fact the velocity of the particle, so it has indeed to be equal to zero in a convergence point.

Example

$$\begin{cases} v_0 = 1, x_0 = 4.5 \\ p_1 = 3, p_2 = 4 \\ \varphi_{\max,1} = 0.1, \varphi_{\max,2} = 5 \end{cases}$$

φ_1 and φ_2 are uniform random variables between 0 and $\varphi_{\max,1}$ and $\varphi_{\max,2}$ respectively.


4,5
4,4
4,3
4,2
4,1
4
3,9
3,8
3,7
x
-1,5
-1
-0,5
0
0,5
1
v