Algebraic view

The basic simplified dynamic system is defined by

$$\begin{cases} v_{t+1} = v_t + \varphi y_t \\ y_{t+1} = -v_t + (1-\varphi) y_t \end{cases} \qquad \textbf{Equ. 1}$$

where $y_t = p - x_t$.

Let $P_t = \begin{bmatrix} v_t \\ y_t \end{bmatrix}$ be the current point in R^2, and $M = \begin{bmatrix} 1 & \varphi \\ -1 & 1-\varphi \end{bmatrix}$ the matrix of the system. So we have $P_{t+1} = MP_t$ and, more generally, $P_t = M^t P_0$

So the system is completely defined by *M*.

The eigenvalues of *M* are:

$$\begin{cases} e_1 = 1 - \dfrac{\varphi}{2} + \dfrac{\sqrt{\varphi^2 - 4\varphi}}{2} \\ e_2 = 1 - \dfrac{\varphi}{2} - \dfrac{\sqrt{\varphi^2 - 4\varphi}}{2} \end{cases} \qquad \textbf{Equ. 2}$$

We see immediately that the value φ=4 is special. We will see below what it means.

For $\varphi \neq 4$ we can define a matrix *A* so that $AMA^{-1} = \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix}$

(if φ =4, A^{-1} doesn't exist).

For example, from the canonical form $A = \begin{bmatrix} a & 1 \\ c & 1 \end{bmatrix}$ we find

$$\begin{cases} a = \dfrac{\varphi + \sqrt{\varphi^2 - 4\varphi}}{2\varphi} \\ c = \dfrac{\varphi - \sqrt{\varphi^2 - 4\varphi}}{2\varphi} \end{cases}$$

In order to have simpler formulas, we can multiply by 2φ, to produce a matrix *A*:

$$A = \begin{bmatrix} \varphi + \sqrt{\varphi^2 - 4\varphi} & \varphi \\ \varphi - \sqrt{\varphi^2 - 4\varphi} & \varphi \end{bmatrix}$$

So if we define $Q_t = AP_t$ we can now write

$$\begin{aligned} P_{t+1} &= A^{-1} LAP_t \\ AP_{t+1} &= LAP_t \\ Q_{t+1} &= LQ_t \end{aligned}$$

that is to say we have, finally, $Q_t = L^t Q_0$

But L is a diagonal matrix, so we have simply $L^t = \begin{bmatrix} e_1^t & 0 \\ 0 & e_2^t \end{bmatrix}$

In particular, we have a cyclic behavior if and only if $Q_t = Q_0$ (or, more generally if $Q_{t+k} = Q_t$). This just means that we have the system of two equations:

$$\begin{cases} e_1^t = 1 \\ e_2^t = 1 \end{cases}$$

Case φ<4

For $\phi<4$, the eigenvalues are complex, and there is always at least one (real) solution for ϕ.
More precisely we can write

$$\begin{cases} e_1 = \cos(\theta) + i\sin(\theta) \\ e_2 = \cos(\theta) - i\sin(\theta) \end{cases}$$

with $\cos(\theta) = 1 - \frac{\phi}{2}$ and $\sin(\theta) = \frac{\sqrt{4\phi - \phi^2}}{2}$
and then

$$\begin{cases} e_1^t = \cos(t\theta) + i\sin(t\theta) \\ e_2^t = \cos(t\theta) - i\sin(t\theta) \end{cases}$$

and cycles are given by any θ so that $\theta = \frac{2k\pi}{t}$

So for each t, the solutions for ϕ are given by

$$\phi = 2\left(1 - \cos\frac{2k\pi}{t}\right), k \in \{1,2,\ldots,t-1\}$$

Table 1 gives some nontrivial values of ϕ for which the system is cyclic.

Table 1. Some ϕ values for which the system is cyclic.

ϕ	***size of the cycle***
3	3 (see Figure 1)
2	4
$\frac{5 \pm \sqrt{5}}{2}$	5 (see Figure 2 and Figure 3)
1, 3	6,3
$1,\ 2,\ 3,\ 2 \pm \sqrt{3}$	6, 4, 3, 12

For any other value, the system is just quasi-cyclic (see Figure 4).
We can be a little bit more precise. Below, $\| \; \|$ is the 2-norm (the Euclidean one for a vector).
We have here

$$\begin{cases} \|Q_t\| = \|AP_t\| = \|Q_0\| \\ \|A^{-1}\| \|Q_0\| \geq \|P_t\| \geq \dfrac{\|Q_0\|}{\|A\|} \end{cases}$$

For example, for v_0=0 and y_0=1, we have

$$\sqrt{\max\left(\frac{1}{2}\left|\frac{3\phi-4\pm\sqrt{5\phi^2-8\phi+16}}{\phi^2-4\phi}\right|\right)}\leq\left\|P_t\right\|\leq\sqrt{\frac{2}{\max\left(\left|3\phi-4\pm\sqrt{5\phi^2-8\phi+16}\right|\right)}}$$

Figure 1. 3-cycle.

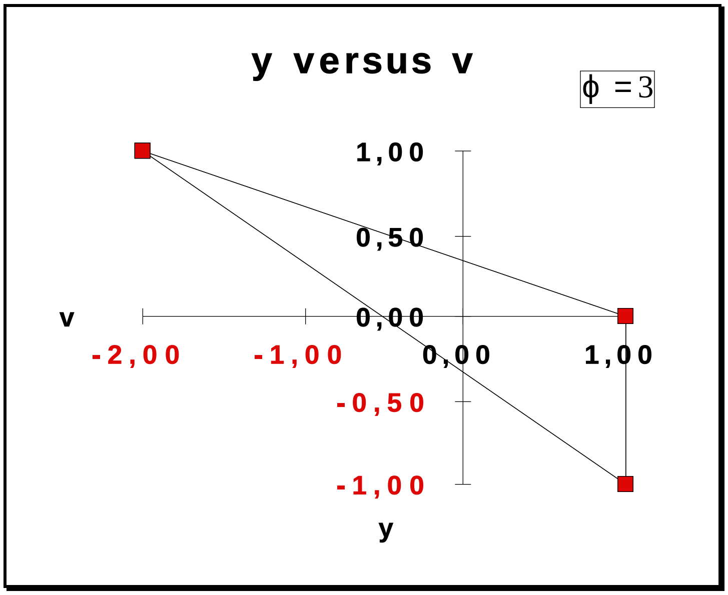


Figure 2. Non convex 5-cycle.

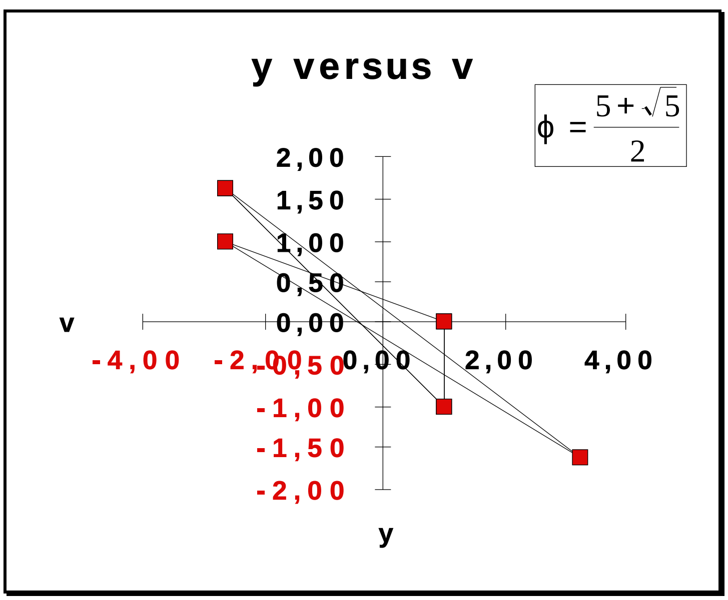


Figure 3. Convex 5-cycle.

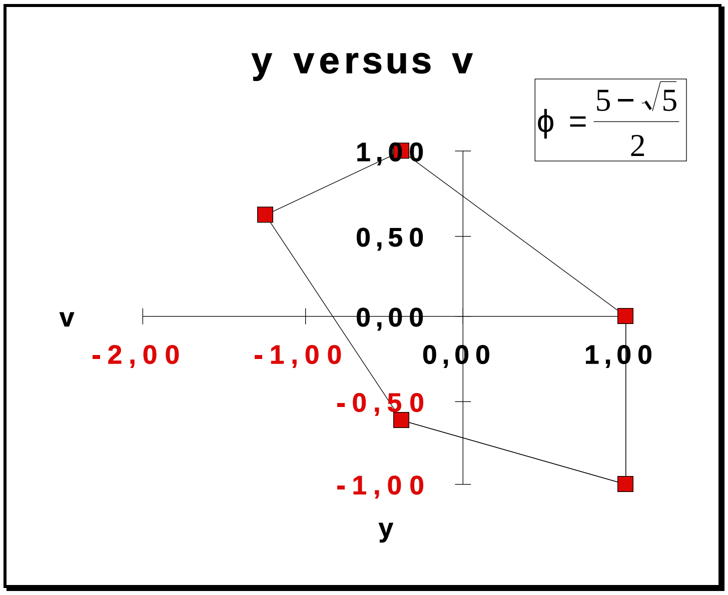
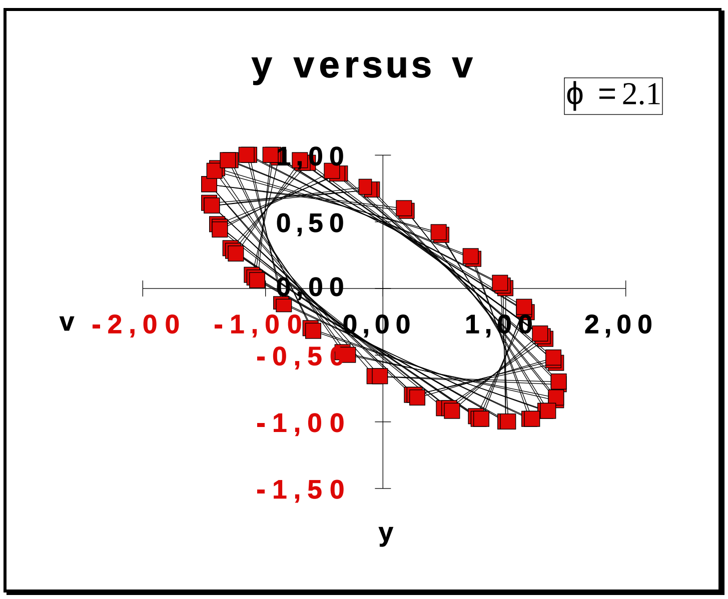


Figure 4. Quasi-cycle.



Case ϕ>4

If ϕ>4, then e_1 and e_2 are real numbers (and $e_1 \le e_2$), so we have either

- $e_1 = e_2 = 1$ (for t even) which implies ϕ=0, not consistent with the hypothesis ϕ>4
- $e_1 = -e_2 = 1$ (or -1) which is impossible
- $e_1 = e_2 = -1$ that is to say ϕ=4, not consistent with the hypothesis ϕ>4

So, and this is the point, there is no cyclic behavior for ϕ>4. And, in fact, the distance from the point P_t to the center (0,0) is strictly increasing with t.

We have

$$Q_t = AP_t$$

$$L^t Q_0 = AP_t$$

$$L^t Q_0 = AP_t$$

So

$$\left\| L^t Q_0 \right\| \le \|A\| \|P_t\|$$

$$\frac{\left\| L^t Q_0 \right\|}{\|A\|} \le \|P_t\|$$

But we can also write

$$P_t = A^{-1} Q_t$$

$$\|P_t\| \le \left\| A^{-1} \right\| \|Q_t\|$$

$$\|P_t\| \le \left\| A^{-1} \right\| \left\| L^t Q_0 \right\|$$

So, finally, $\|P_t\|$ is increasing “like” $\left\| L^t Q_0 \right\|$.

This result can be used to prevent the “explosion” of the system by defining "constriction" coefficients.

Case φ=4

We have here $M = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$

In this particular case, the eigenvalues are both equal to -1, and there is just one family of eigenvectors, generated by $V = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. So we have $MV = -V$.

So, if P_0 is an eigenvector, proportional to V (that is to say if $v_0 + 2y_0 = 0$), we just have two “symmetrical” points, for

$$P_{t+1} = \pm \begin{bmatrix} 2y_0 \\ -y_0 \end{bmatrix} = -P_t$$

In the case where P_0 is not an eigenvector, we compute directly how $\|P_t\|$ is decreasing and/or increasing.

Let us define $\Delta_t = \|P_{t+1}\|^2 - \|P_t\|^2$.

It is easy to see (by recurrence) we have has the following form:

$$\Delta_t = a_t v_0^2 + b_t v_0 y_0 + c_t y_0^2$$

where a_t, b_t, c_t are integer numbers so that $\Delta_t = 0$ for $v_0 + 2y_0 = 0$.

Now, let’s suppose for a particular t we have $\Delta_t > 0$. What about Δ_{t+1} ?

We easily compute $\Delta_t = v_t^2 + 14 v_t y_t + 24 y_t^2$.

This quantity is positive if and only if v_t is not between (or equal to) the roots $\{-2y_t, -12y_t\}$

Now, if we compute Δ_{t+1} we have $\Delta_{t+1} = 11 v_t^2 + 54 v_t y_t + 64 y_t^2$, and the roots are $\left\{-2y_t, -\frac{32 y_t}{11}\right\}$. As $\frac{32}{11} < 12$, it means that Δ_{t+1} is also positive.

So as soon as $\|P_t\|$ begins to increase, it does so infinitely.

But it can be decreasing, at the beginning. How many times ?

Suppose we have $\Delta_0 < 0$.

It means v_0 is between $-2y_0$ and $-12y_0$. For instance in the case $y_0>0$, we can write

$v_0 = -2y_0 - \varepsilon$, with $\varepsilon \in \left]0,10y_0\right[$

By recurrence, we have then

$$\Delta_0 = -10y_0\varepsilon + \varepsilon^2$$

$$\Delta_1 = -10y_0\varepsilon + 11\varepsilon^2$$

$$\Delta_2 = -10y_0\varepsilon + 21\varepsilon^2$$

$$\Delta_{t+2} = -\Delta_t + 2\Delta_{t+1} = -10y_0\varepsilon + k_{t+2}\varepsilon^2 \text{, with } k_{t+2} = -k_t + 2k_{t+1}$$

Finally, we can write

$$\Delta_t = -10y_0\varepsilon + (1+10t)\varepsilon^2$$

as long as

$$(1+10t)\varepsilon^2 \le 10y_0\varepsilon$$

that is to say (for t is an integer) as long as

$$t \le 1 + Integer\_part\left(\frac{y_0}{\varepsilon}\right)$$

After that, $\|P_t\|$ increases.

We can do exactly the same analysis for $y_0<0$. In this case $\varepsilon<0$ too, so the formula is the same.

In fact, we can even be more precise. If we define

$$\begin{cases} \alpha = -10y_0\varepsilon + \varepsilon^2 \\ \beta = 10\varepsilon^2 \end{cases}$$

then we have

$$\|P_t\| = t\sqrt{\frac{\beta}{2} + \frac{\alpha - \frac{\beta}{2}}{t} + \frac{\|P_0\|^2}{t^2}}$$

That is to say $\|P_t\|$ is decreasing/increasing almost linearly when t is big enough. In particular, even if it begins to decreases, after that it tends to increase almost like $t\sqrt{5}\left|v_0 + 2y_0\right|$.